

# Representations and classification of traveling wave solutions to Sinh-Gördon equation

Cheng-shi Liu  
 Department of Mathematics  
 Daqing Petroleum Institute  
 Daqing 163318, China  
 Email: chengshiliu-68@126.com

February 1, 2008

## Abstract

Two concepts named atom solution and combinatory solution are defined. The classification of all single traveling wave atom solutions to Sinh-Gördon equation is obtained, and qualitative properties of solutions are discussed. In particular, we point out that some qualitative properties derived intuitively from dynamic system method aren't true. In final, we prove that our solutions to Sinh-Gördon equation include all solutions obtained in the paper[Fu Z T et al, Commu. in Theor. Phys.(Beijing) 2006 45 55]. Through an example, we show how to give some new identities on Jacobian elliptic functions.

Keywords: traveling wave solution, atom solution, exact solution, Sinh-Gördon equation, elliptic function

PACS: 05.45.Yv, 03.65.Ge, 02.30.Jr

## 1 Introduction

The construction of traveling wave solutions for nonlinear evolution equations is one of important aims in nonlinear science. Various methods are proposed for this purpose (for example, see [1-28, 30-35]). By these methods, a large number of nonlinear equations are solved and their abundant traveling wave solutions are obtained. Although these methods give some applicable results, some problems are also need further studied. For example, can classification of all single traveling wave solutions be given? How to construct new solutions? How to decide that a solution is novel? In order to solve these problem, we first give the following definitions.

**Definition 1:** For a given nonlinear evolution, if its one traveling wave solution is obtained by direct integral method, we call this solution an atom solution.

**Definition 2:** A solution constructed piecewise in terms of atom solutions is called a combinatory solution.

For these equations which can be directly reduced to an elementary integral form, we can obtain classifications of their all single traveling wave atom solutions. Based on these atom solutions, by piecewise construction, we can give some combinatory solutions with some new properties, such as peakon solutions[27], compacton solutions[28], etc. Therefore, to give classification of all atom solutions is an important aim. In my early papers[7, 26], I have obtained classifications of atom solutions for some nonlinear equations, in which I don't define the concept of atom solution. Hence, here I must point out that it is just for atom solutions to give those corresponding classifications. In present paper, I discuss Sinh-Gördon equation. It reads

$$u_{xt} = \alpha \sinh u, \quad (1)$$

which is applied to integrable quantum field, noncommutative field theories, fluid dynamics, geometry [29], etc. Its some exact solutions have been obtained by some authors [30, 31, 32, 33, 34, 35]. In order to solve its new single traveling wave solutions, Fu *et al* [30] try to introduce two variable transformations, and obtain some special form solutions under some special cases. But we must point out that it is a rather easy thing to obtain all single traveling wave atom solutions to Sinh-Gördon equation. Using elementary integral method and complete discrimination system for polynomial[5-8], we obtain this complete result. On the other hand, from qualitative analysis, intuitively, no periodic solution exists for Sinh-Gördon equation. But we have several periodic solutions indeed. Hence it seems that there is a contradictory. We discuss this problem and show that sometimes qualitative conclusion may be not true.

Some authors don't believe that these solutions obtained by our method are all atom solutions of Sinh-Gördon equation. They point out to me that professors Liu Shi-Kuo *et al* [36, 37] have given a lot of different solutions to elliptic equation, and hence ask me how I can transfer their solutions into our solutions. In the present paper, I prove this conclusion. In other words, their solutions only give some new representations of solutions to elliptic equation rather than new solution. In particular, I must emphasize that my proofs are not direct. Furthermore, from these proofs, we can obtain some interesting new identities on Jacobian elliptic functions. Through an example, we show how to do this thing.

This paper is organized as follows: In Section 2, we give the classification of all traveling wave atom solutions to Sinh-Gördon equation. In Section 3, we discuss qualitative analysis of solutions for Sinh-Gördon equation. In Section 4, we prove that all solutions to Sinh-Gördon equation given in reference [30] can be represented by our solutions. In particular, through an example, we show how to give some new identities on Jacobian elliptic functions. The last section is a short summary.

## 2 Classification of traveling wave atom solutions to Sinh-Gördon equation

We take traveling wave transformation

$$u = u(\xi), \quad \xi = kx + \omega t. \quad (2)$$

Substituting Eq.(2) into Eq.(1) yields

$$k\omega u'' = \alpha \sinh u. \quad (3)$$

By integrating Eq.(2) once and rewriting it as integral form, we have

$$\pm (\xi - \xi_0) = \int \frac{du}{\sqrt{\frac{2\alpha}{k\omega}(\cosh u + c)}}. \quad (4)$$

It is rather natural to take variable transformation as follows

$$w = \exp u. \quad (5)$$

Then  $w > 0$  and Eq.(4) becomes

$$\pm (\xi - \xi_0) = \int \frac{1}{\sqrt{\frac{\alpha}{k\omega}w(w^2 + 2cw + 1)}} dw. \quad (6)$$

Denote  $\Delta = 4c^2 - 4$ . If  $\frac{\alpha}{k\omega} > 0$ , there are the following four cases to be discussed.

Case 1:  $\Delta = 0$ , that is,  $c = \pm 1$ . Since  $w > 0$ , if  $\frac{\alpha}{k\omega} < 0$ , then no solutions to Eq.(6). When  $\frac{\alpha}{k\omega} > 0$ , there are the following two cases

If  $c = -1$ , we have

$$u = \ln \tanh^2\left(\frac{1}{2}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)\right), \quad (7)$$

$$u = \ln \coth^2\left(\frac{1}{2}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)\right). \quad (8)$$

If  $c = 1$ , we have

$$u = \ln \tan^2\left(\frac{1}{2}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)\right). \quad (9)$$

Case 2:  $\Delta > 0$ . Suppose that  $\rho < \beta < \gamma$ , one of them is zero, and others two roots of  $w^2 + 2cw + 1 = 0$ . For  $\frac{\alpha}{k\omega} > 0$ , when  $\rho < w < \beta$  and  $w > 0$ , we have

$$u = \pm \ln\{\rho + (\beta - \rho) \operatorname{sn}^2\left(\frac{\sqrt{\gamma - \rho}}{2}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0), m\right)\}. \quad (10)$$

Concretely, the solutions are given by

$$u = \pm \ln\{m \operatorname{sn}^2(\frac{1}{2\sqrt{m}}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)\}. \quad (11)$$

where  $0 < m < 1$ .

When  $w > \gamma$  and  $w > 0$ , we have

$$u = \pm \ln\left\{\frac{\gamma - \beta \operatorname{sn}^2(\frac{\sqrt{\gamma-\rho}}{2}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)}{\operatorname{cn}^2(\frac{\sqrt{\gamma-\rho}}{2}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)}\right\}, \quad (12)$$

where  $m^2 = \frac{\beta-\rho}{\gamma-\rho}$ . Concretely, the solutions are given by

$$u = \pm \ln\left\{\frac{1 - m^2 \operatorname{sn}^2(\frac{1}{2\sqrt{m}}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)}{m \operatorname{cn}^2(\frac{1}{2\sqrt{m}}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)}\right\}, (c < -1), \quad (13)$$

and

$$u = \pm \ln\left\{\frac{m \operatorname{sn}^2(\frac{1}{2\sqrt{m}}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)}{\operatorname{cn}^2(\frac{1}{2\sqrt{m}}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)}\right\}, (c > 1). \quad (14)$$

For  $\frac{\alpha}{k\omega} < 0$ , when  $\beta < w < \gamma$  and  $w > 0$ , we have

$$u = \pm \ln\left\{\beta - (\beta - \gamma) \operatorname{sn}^2(\frac{\sqrt{\gamma-\rho}}{2}\sqrt{-\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)\right\}, \quad (15)$$

where  $m^2 = \frac{\beta-\rho}{\gamma-\rho}$ . Concretely, we have

$$u = \pm \ln\left\{\frac{1 - (1 - m^2) \operatorname{sn}^2(\frac{1}{2\sqrt{m}}\sqrt{-\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)}{m \operatorname{cn}^2(\frac{1}{2\sqrt{m}}\sqrt{-\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)}\right\}, (c < -1). \quad (16)$$

**Remark 1:** Because two roots of the equation  $w^2 + 2cw + 1 = 0$  have the same signs, we have  $\rho = 0$  or  $\gamma = 0$ . Moreover, the product of two nonzero roots is 1. These facts are important for verifying the above solutions.

Case3:  $\Delta < 0$ . When  $\frac{\alpha}{k\omega} < 0$ , no solutions to Eq.(6). When  $\frac{\alpha}{k\omega} > 0$ , we have

$$u = \pm \ln\left\{\frac{2}{1 + \operatorname{cn}(\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0)), m)} - 1\right\}, \quad (17)$$

where  $m^2 = \frac{1}{2}(1 - c)$ .

Expressions (7-9),(11)(13)(14)(16)(17) are all solutions to Eq.(4). Thus we give classification of all solutions to Eq.(3). These solutions are all possible single traveling wave atom solutions to Sinh-Gördon equation. This is a complete result.

**Remark 2:** It is easy to see that if  $u$  is a solution of Sinh-Gördon equation, then  $-u$  is also a solution. For example,  $u = \ln \tan^2(\frac{1}{2}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0))$  is a solution, then  $u = \ln \cot^2(\frac{1}{2}\sqrt{\frac{\alpha}{k\omega}}(\xi - \xi_0))$  is also a solution. But we can transfer these two solutions each other by taking  $\xi_0$  as  $\xi_0 + \frac{\pi}{2}$ . For convenience, we add a sign  $\pm$  in front of the expressions (10)-(17).

### 3 Qualitative analysis

Let us analysis qualitative properties of the above traveling wave solutions of Sinh-Gördon equation. Here we use dynamic system method. Rewrite sinh-Gördon equation as

$$u' = v, \quad (18)$$

$$v' = \frac{\alpha}{k\omega} \sinh u. \quad (19)$$

Its unique singular point is  $(0, 0)$ , in which eigenvalue of Jacobian matrix satisfies  $\lambda^2 = \frac{\alpha}{k\omega}$ . Therefore, for  $\frac{\alpha}{k\omega} > 0$ ,  $(0, 0)$  is a saddle point, and for  $\frac{\alpha}{k\omega} < 0$ ,  $(0, 0)$  is a center point. Furthermore, according to the theory of dynamic system, it seems that it is impossible to give periodic solution to sinh-Gördon equation for  $\frac{\alpha}{k\omega} > 0$ . But in fact solution (9) is a periodic solution, and solutions (13) and (14) are the double periodic solutions. This is a contradictory. Why? How to solve this problem?

It is easy to prove that expressions (9), (13) and (14) are indeed the solutions of Sinh-Gördon equation through substituting directly these solution into it. So we have no doubt for these solutions. If we notice that solution (9) is discontinuous periodic function, then we can conclude that there isn't contradictory at all because that so-called periodic solution needed for saddle point must be continuous function. Based on the same reason, solutions (13), (14) and (17) are also reasonable.

In fact, for example, the solution (9) has period  $\pi$  and discontinuous points  $k\pi/2$  with  $k$  arbitrary integer, and it is unbounded and monotonic in  $(-\pi/2 + k\pi, k\pi)$  or  $(k\pi, k\pi + \pi/2)$ , etc. If starting point is given, then the trajectory will move continuously along a branch in a periodic interval, and can't move from one periodic interval to another periodic interval. Thus these so-called periodic solutions are not realizable periodic solutions. We give this kind of solution a name by

**Definition 3:** Nearby a saddle point, if a solution is a discontinuous periodic solution, then we call it a formal-periodic solution.

Therefore, solutions (9), (13), (14), (16) and (17) are formal-periodic solutions to Sinh-Gördon equation.

A problem need studied further is when a formal-periodic solution exists.

## 4 Representations of solutions

In Ref.[30], Fu *et al* introduce two transformations

$$u = 2 \sinh^{-1} v \quad (20)$$

and

$$u = 2 \cosh^{-1} v \quad (21)$$

to try to obtain new solutions to Sinh-Göordon equation. But if we write these two transformations as more explicit forms

$$u = 2 \ln(v + \sqrt{v^2 + 1}) \quad (22)$$

and

$$u = 2 \ln(v \pm \sqrt{v^2 - 1}), \quad (23)$$

we can see that these transformations are rather complicated. However, for example, if we take

$$w = (v + \sqrt{v^2 + 1})^2, \quad (24)$$

then we have

$$v = \frac{w - 1}{2\sqrt{w}}, \quad (25)$$

and hence we can transfer those results in Ref.[30] into our expressions. According to our method used here, we can prove that all solutions in Ref.[30] are the special cases of our solutions. For shortly, we only consider the first kind transformation, i.e.,  $u = 2 \sinh^{-1} v$ , and the first solutions  $u_1$  in Ref.[30]. The second kind transformation and other solutions can be dealt with similarly. In order to illustrate my method, as an example, we solve the solutions of  $(u')^2 = 1 - u^2$ . Obviously, its solutions are

$$u(\xi) = \pm \sin(\xi - \xi_0). \quad (26)$$

On the other hand, if we take transformation of variable

$$u = \frac{1 - w^2}{1 + w^2}, \quad (27)$$

then corresponding integral is

$$\pm (\xi - \xi_1) = -2 \int \frac{dw}{1 + w^2} = -2 \arctan w. \quad (28)$$

Hence the solutions are given by

$$w = \pm \tan\left(\frac{1}{2}(\xi - \xi_1)\right), \quad (29)$$

that is

$$u(\xi) = \frac{1 - \tan^2\left(\frac{1}{2}(\xi - \xi_1)\right)}{1 + \tan^2\left(\frac{1}{2}(\xi - \xi_1)\right)} = \cos(\xi - \xi_1). \quad (30)$$

If we take  $\xi_1 = \xi_0 + \frac{\pi}{2}$ , then

$$\sin(\xi - \xi_0) = \cos(\xi - \xi_1); \quad (31)$$

If we take  $\xi_1 = \xi_0 + \frac{3\pi}{2}$ , then

$$-\sin(\xi - \xi_0) = \cos(\xi - \xi_1). \quad (32)$$

This show that these two solutions in distinct forms are indeed the same solution with distinct integral constants.

Now we consider the following elliptic equation

$$(v')^2 = (v^2 + 1)(v^2 - c), \quad (33)$$

where we take  $c > 0$ . We use two transformations to solve the corresponding integral so that we obtain a solution which has two distinct forms. Firstly, by taking the transformation

$$v = \pm\sqrt{y}, \quad (34)$$

we have

$$\pm 2(\xi - \xi_0) = \int \frac{dy}{\sqrt{y(y+1)(y-c)}}. \quad (35)$$

When  $y > c$ , the corresponding solution is

$$y = \frac{c}{\text{cn}^2(\sqrt{c+1}(\xi - \xi_0), m_1)}, \quad (36)$$

where  $m_1^2 = \frac{1}{1+c}$ . The corresponding  $v$  is

$$v = \pm \frac{\sqrt{c}}{\text{cn}(\sqrt{c+1}(\xi - \xi_0), m_1)}. \quad (37)$$

Secondly, by taking the transformation

$$v = \frac{w-1}{2\sqrt{w}}, \quad (38)$$

where  $w > 0$ , integral becomes

$$\pm(\xi - \xi_1) = \int \frac{dw}{\sqrt{w(w^2 - 2(2c+1)w + 1)}}. \quad (39)$$

Since  $v^2 > c$ , we have  $w > 2c+1+2\sqrt{c(c+1)}$ , and hence the corresponding solution  $w$  is given by

$$w = \frac{2c+1+2\sqrt{c(c+1)} - (2c+1-2\sqrt{c(c+1)}) \text{sn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi - \xi_1), m_2)}{\text{cn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi - \xi_1), m_2)}, \quad (40)$$

where  $m_2^2 = \frac{2c+1-2\sqrt{c(c+1)}}{2c+1+2\sqrt{c(c+1)}}$ . Therefore, we have

$$v = \frac{\frac{2c+1+2\sqrt{c(c+1)}-(2c+1-2\sqrt{c(c+1)}) \operatorname{sn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}{\operatorname{cn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)} - 1}{2\sqrt{\frac{2c+1+2\sqrt{c(c+1)}-(2c+1-2\sqrt{c(c+1)}) \operatorname{sn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}{\operatorname{cn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}}. \quad (41)$$

When  $\xi = \xi_1$ , we have

$$w(\xi_1) = 2c + 1 + 2\sqrt{c(c+1)}, \quad (42)$$

that is

$$v(\xi_1) = \frac{2c + 2\sqrt{c(c+1)}}{2\sqrt{2c + 1 + 2\sqrt{c(c+1)}}}, \quad (43)$$

and hence we have

$$\pm \frac{\sqrt{c}}{\operatorname{cn}(\sqrt{c+1}(\xi_1 - \xi_0), m_1)} = \frac{2c + 2\sqrt{c(c+1)}}{2\sqrt{2c + 1 + 2\sqrt{c(c+1)}}}. \quad (44)$$

Thus we conclude that if

$$\operatorname{cn}(\sqrt{c+1}(\xi_1 - \xi_0), m_1) = \frac{\sqrt{2c + 1 + 2\sqrt{c(c+1)}}}{1 + \sqrt{c+1}}, \quad (45)$$

then

$$\begin{aligned} v &= \frac{\sqrt{c}}{\operatorname{cn}(\sqrt{c+1}(\xi - \xi_0), m_1)} \\ &= \frac{\frac{2c+1+2\sqrt{c(c+1)}-(2c+1-2\sqrt{c(c+1)}) \operatorname{sn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}{\operatorname{cn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)} - 1}{2\sqrt{\frac{2c+1+2\sqrt{c(c+1)}-(2c+1-2\sqrt{c(c+1)}) \operatorname{sn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}{\operatorname{cn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}}; \end{aligned} \quad (46)$$

if

$$\operatorname{cn}(\sqrt{c+1}(\xi_1 - \xi_0), m_1) = -\frac{\sqrt{2c + 1 + 2\sqrt{c(c+1)}}}{1 + \sqrt{c+1}}, \quad (47)$$

then

$$v = -\frac{\sqrt{c}}{\operatorname{cn}(\sqrt{c+1}(\xi - \xi_0), m_1)}$$



$$\begin{aligned}
& \frac{2c+1+2\sqrt{c(c+1)}-(2c+1-2\sqrt{c(c+1)}) \operatorname{sn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}{\operatorname{cn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)} - 1 \\
&= \frac{\frac{2c+1+2\sqrt{c(c+1)}-(2c+1-2\sqrt{c(c+1)}) \operatorname{sn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}{\operatorname{cn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}}{2\sqrt{\frac{2c+1+2\sqrt{c(c+1)}-(2c+1-2\sqrt{c(c+1)}) \operatorname{sn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}{\operatorname{cn}^2(\frac{\sqrt{2c+1+2\sqrt{c(c+1)}}}{2}(\xi-\xi_1), m_2)}}}. \quad (48)
\end{aligned}$$

According to the method and the above results, it is easy to prove that all solutions obtained in Ref.[30] can be represented by our solutions. For shortly, we only consider the solution  $u_1$  in Ref.[30], which has the following form

$$u_1 = 2 \sinh^{-1}(\pm \frac{m}{\sqrt{1-m^2}} \operatorname{cn} \xi), \quad (49)$$

where  $0 < m < 1$ , and  $\frac{\alpha}{k^2 c} = -\frac{\alpha}{k\omega} = 1-m^2$ . Other cases can be proven similarly, we omit them here.

In fact, if we take traveling wave transformation  $\xi = k(x-ct)$ , then  $\omega = -kc$ . Under the first transformation of variable in Ref.[30], the corresponding ordinary equation is given by

$$(1+v^2)v'' - v(v')^2 + \alpha_1 v(1+v^2)^2 = 0, \quad (50)$$

where  $\alpha_1 = \frac{\alpha}{k^2 c}$ . Its general solution is

$$\pm(\xi - \xi_0) = \int \frac{dv}{\sqrt{-\alpha_1(v^2+1)(v^2-c_2)}}, \quad (51)$$

where  $c_2$  is an integral constant. We take

$$v = \frac{w-1}{2\sqrt{w}}, \quad c_1 = -(1+2c_2), \quad (52)$$

then corresponding equation is

$$\pm(\xi - \xi_1) = \int \frac{dw}{\sqrt{-\alpha_1 w(w^2+2c_1 w+1)}}. \quad (53)$$

We only consider the case  $\alpha_1 > 0$ , and the case  $\alpha_1 < 0$  can be dealt with similarly. Then we have  $c_2 > v^2 > 0$ , and hence  $c_1 < -1$ . From  $-\sqrt{c_2} < v < \sqrt{c_2}$ , we have

$$c_1 - \sqrt{c_1^2 - 1} < -w < c_1 + \sqrt{c_1^2 - 1}. \quad (54)$$

Therefore the solution is

$$w = -c_1 + \sqrt{c_1^2 - 1} - 2\sqrt{c_1^2 - 1} \operatorname{sn}^2(\frac{\sqrt{\alpha_1(\sqrt{c_1^2 - 1} - c_1)}}{2}(\xi - \xi_0), m_1), \quad (55)$$

where  $m_1^2 = \frac{2\sqrt{c_1^2-1}}{\sqrt{c_1^2-1}-c_1}$ . In Ref.[30], when  $\alpha_1 = 1 - m^2$  and  $c_2 = \frac{m^2}{1-m^2}$ , Fu *et al* give solution

$$v_1 = \pm \frac{m}{\sqrt{1-m^2}} \text{cn}(\xi, m). \quad (56)$$

In order to remove '±', we take  $\xi_1$  such that  $\text{cn}(\xi_1, m) = \pm 1$ , and hence the above solution  $v_1$  can be rewritten as

$$v_1 = \frac{m}{\sqrt{1-m^2}} \text{cn}(\xi - \xi_1, m). \quad (57)$$

Correspondingly, we have  $c_1 = -\frac{1+m^2}{1-m^2}$ . From  $w_0 = w(\xi_0) = \frac{(1+m)^2}{1-m^2}$ , it follows that  $v(\xi_0) = \frac{w_0-1}{2\sqrt{w_0}} = \frac{m}{\sqrt{1-m^2}}$ . On the other hand, we have  $v(\xi_0) = \frac{m}{\sqrt{1-m^2}} \text{cn}(\xi_0 - \xi_1, m)$ , so  $\text{cn}(\xi_0 - \xi_1, m) = 1$ . Under this condition, according to  $w = (v + \sqrt{1+v^2})^2$ , we obtain a new identity of elliptic functions

$$\begin{aligned} & \frac{(1+m)^2}{1-m^2} - \frac{4m}{1-m^2} \text{sn}^2\left(\frac{1+m}{2}(\xi - \xi_0), \frac{4m}{(1+m)^2}\right) \\ &= \left\{ \frac{m}{\sqrt{1-m^2}} \text{cn}(\xi - \xi_1, m) + \sqrt{1 + \frac{m^2}{1-m^2} \text{cn}^2(\xi - \xi_1, m)} \right\}^2. \end{aligned} \quad (58)$$

This formula also give a new representation of solutions to Sinh-Gördon equation, i.e.,

$$u_1 = \ln w = \ln(v + \sqrt{1+v^2})^2 = 2 \sinh^{-1} v. \quad (59)$$

According to the above method, other 36 solutions in Ref.[30] can be represented by our solutions and furthermore obtain corresponding new identities on Jacobian elliptic functions. For shortly, we don't write those formulae concretely. In final, I would like to point out that, in Ref.[38-40], by other transformations like Landen's quadratic transformation, some identities on Jacobian functions have been given. But Our results are novel.

## 5 Conclusions

We obtain easily all single traveling wave atom solutions to Sinh-Gördon equation using direct integral method. Qualitative properties of solutions are discussed, and some intuitive conclusions are corrected. In order to prove that all other forms of solutions in Ref.[30] can be represented by our solutions, we discuss the solutions to elliptic equation. As a result, we can give some new identities on Jacobian elliptic functions and show that all solutions to Sinh-Gördon equation in Ref.[30] can be derived from our solutions. By some examples, we display our method and results.

**Acknowledgements:** I would like to thank referee for his (or her) suggestion for discussing qualitative properties of solutions to Sinh-Gördon equation so that I can clarify those contradictories.

## References

- [1] Wadati M, Konno K and Ichikawa Y H. *J. Phys. Soc. Japan*, 1979; 46: 1965
- [2] Wadati and Sawada M K . *J. Phys. Soc. Japan*, 1980; 48: 312
- [3] Wadati M and Toda M. *J. Phys. Soc. Japan*, 1972; 32: 1403
- [4] Wadati M and Ohkuma K. *J. Phys. Soc. Japan*, 1982;51:2029
- [5] Liu C S and Du X H. *Acta. Phys. Sin.* 2005; 54: 1709
- [6] Liu C S. *Chin. Phys. Lett.* 2004; 21: 1369
- [7] Liu C S. *Chin. Phys.* 2005; 14: 1710
- [8] Liu C S. *Comm. Theor. Phys.* 2005; 43: 787
- [9] Ablowitz M J and Clarkson P A. Solitons, nonlinear evolutions and inverse scattering.;Cambridge: Cambridge University Press. 1991
- [10] Miura R M. ed. Bachlund transformation. Lecture notes in Mathematics 515. New York:Springer-Verlag. 1976
- [11] Hirota R. Direct method in soliton theory. In Solitons. Eds. Bullough R K and Caudrey P J, Topics in current Physics 17, 157-176, New York Springer Verlag 1980
- [12] Wang M L,Zhou Y B.,Li Z B. *Phy. Lett. A*. 1996; 216: 67
- [13] Wang M L. *Phys. Lett. A*. 1995; 199: 169
- [14] Fan E G. *Phys.Lett* . 2000; A277: 212.
- [15] Liu S K, Fu Z T, Liu S D, Zhao Q. *Phys. Lett. A* 2001; 290: 72.
- [16] Yan Z Y .*Chaos,Solitons and Fractals*. 2003; 15: 575
- [17] Lou S Y, Lu J Z. *J. Phys. A*. 1996; 29: 4029
- [18] Fan E G. *J. Phys. Soc. Japan*. 2002; 71: 2663.
- [19] Fan E G. *Chaos, Solitons and Fractals*. 2003;16: 819
- [20] Fan E G. *J. Phys. A*. 2002; 35: 6853
- [21] Fan E G. *J. Phys. A*. 2000; 33: 6925
- [22] Liu C S. *Acta.Phys. Sin.* 2005; 54: 2505(in Chinese)
- [23] Liu C S.*Commun. in Theor.Phys.* 2005; 44: 799
- [24] Liu C S.*Commun. in Theor.Phys.* 2006; 45: 219

- [25] Liu C S. *Commun. in Theor. Phys.* 2005; 45: 395
- [26] Liu C S. *Commun. in Theor. Phys.* 2005; 45: 991
- [27] Camassa B and Holm D D. *Phys. Rev. Lett.* 1993; 71: 1661
- [28] Rosenau P and Hyman M. *Phys. Rev. Lett.* 1993; 70: 564
- [29] S.S. Chern, *Ann. Polon. Math.* 1980; 39: 74
- [30] Fu Z T et al. *Commu. Theor. Phys.* 2006; 45: 55
- [31] V.E. Chelnokov and M.G. Zeitlin, *Phys. Lett. A* 1983; 99
- [32] A. K. Pogrebkov, *Letters in Mathematical Physics*, 1981; 5: 277
- [33] U. Abresch, *Old and new periodic solutions of the sinh-Gordon equation, Seminar. on new results in non-linear partial differential equations*, Vieweg, Wiesbaden 1987
- [34] Xie Y X and Tang J S, *Il Nuovo Cimento B*, 2006; 121: 115
- [35] Wazwaz Abdul-Majid , *Applied mathematics and computation* 2006; 177: 755
- [36] Fu Z T, Zhang L, Liu S D and Liu S K. *Phys. Lett. A* 2004; 325: 363
- [37] Liu S K and Liu S D. *Nolinear differential equations in physics*. Peking Univ. Press, Beijing, 2000.
- [38] Khare A and Sukhatme U. *Physical Review Letters*. 2002; 88, 244101
- [39] Khare A and Sukhatme U. *Journal of Mathematical Physics*. 2003; 44: 1822
- [40] Khare A and Sukhatme U. arXiv:math-ph/0204054v1, 2002